The p=0 condensate is a myth

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Abstract

Analyzing some of the basic aspects of the dynamics of two bosons (interacting through a central force) and their importance in determining the ground state of a system like liquid ^4He, it is unequivocally concluded that our conventional belief in the existence of $p = 0$ condensate in the superfluid state of such systems [including the state of Bose Einstein condensate (BEC) of trapped dilute gases] is a myth.

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1. Introduction

Over the last seven decades, numerous efforts have been made to develop the microscopic theory of a *system of interacting bosons* (SIB) like liquid ^4He by using different mathematical tools such as variational principle or perturbation technique clubbed with the *most popular presumption* that there exists zero momentum ($p = 0$) condensate in the single particle state in its superfluid phase (see recent articles [1-5] for other important reviews and research papers published on the subject after Bogoliubov's theory [6] of weakly interacting bosons). Nevertheless it is clear that our microscopic understanding of superfluidity of a SIB based on the said $p = 0$ condensate is still open to questions. This is corroborated by the fact that more recently [7], efforts have also been made to develop an alternative theory by introducing coherent pair state (*analogous to the Cooper condensate in a Fermi liquid with attraction between fermions*) along with the existence of $p = 0$ condensate. Guided by these facts we tried to develop another alternative theory [4,5] of such a SIB by obtaining solutions of N-particle Schrödinger equation *without making any ad hoc assumption such as the existence of $p = 0$ condensate* in its superfluid state. Our theory concludes that: (i) each particle in the system represents a pair of particles moving with equal and opposite momenta ($\mathbf{q}$, -$\mathbf{q}$) with respect to their *center of mass* (CM) which moves in the laboratory frame with momentum $\mathbf{K}$ [see Section 2.1 below for detailed meaning of $\mathbf{q}$ and $\mathbf{K}$], (ii) the onset of the superfluid transition represents the *Bose Einstein condensation* (BEC) of these particles (as the representatives of the pair) in a state of $K = 0$ and $q = q_o = \pi/d$ and the system moves from its disordered state in the phase space defined by $\Delta\phi \geq 2\pi$ (with $\Delta\phi$ representing the relative phase position of two neighboring particles) to an ordered state defined by $\Delta\phi = 2n\pi$ ($n = 1, 2, 3, ...$) shifting particles from their random locations (in position space) in the high temperature phase to a close packed arrangement of their *wave packets* in low temperature phase keeping every two neighboring particles at an identically equal distance $d = (V/N)^{1/3}$ (with V being the total volume occupied by N particles); it allows particles to move only in the order of their locations (*obviously with no collision or relative motion leading to zero viscosity*). The theory explains superfluidity and related properties of liquid ^4He to a very good accuracy at quantitative level [4, 8]. Based on *pair of particles basis* (PPB), our theory differs significantly from conventional theories based on *single particle basis* (SPB). For example, the *ground state* (G-state) of the system, whose correct understanding has a key role in revealing the physical behavior of the system, is found to have following important differences, -identified from the results of SPB and PPB.

G-state(SPB) : Different number of atoms have different momenta with certain percentage (*e.g.*, $\approx 10\%$ ^4He atoms in superfluid ^4He) occupying the state of $p = 0$.

G-state(PPB) : Each of the $N(N-1)/2$ pairs, that we can count by way of choosing any two of N particles, has CM momentum $|\mathbf{K}| = 0$ and each atom in the pair, having localized position within the cavity of size d left by its neighbors for its exclusive occupancy, has $q_o = \pi/d$ as the least possible value of q (with $2\mathbf{q} = \mathbf{k}$ being the relative momentum of two particles).

The momentum distribution of particles in G-state(SPB) and G-state(PPB) [5] of a SIB and that of a *system of non-interacting bosons* (SNIB) is depicted in Fig.1 for a better understanding of the difference between these G-states. It is natural that the results of only one (either SPB or PPB) can be correct and to this effect the following analysis unequivocally reveals that only G-state(PPB) concluded by us [4,5] represents the true G-state of a SIB.

2. Dynamics of two particles and G-state of a SIB

2.1. Values of K and q in the G-state

(a). It is well known that the dynamics of two particles (*say*, P1 and P2 interacting through a two body central force), moving with momenta $\mathbf{k}_1$ and $\mathbf{k}_2$ in the laboratory frame, can always be described in terms of their relative momentum $\mathbf{k} = 2\mathbf{q} = \mathbf{k}_2 - \mathbf{k}_1$, and CM momentum $\mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2$, as we have

$$\mathbf{k}_1 = -\mathbf{q} + \mathbf{K}/2, \tag{1}$$

and

$$\mathbf{k}_2 = \mathbf{q} + \mathbf{K}/2. \tag{2}$$

(b). Since CM motion of P1 and P2 *does not encounter inter-particle interaction*, it represents a freely moving body of mass $2m$ implying that the G-state of the pair would invariably have $|\mathbf{K}| = 0$.

(c). With $|\mathbf{K}| = 0$ in Eqns.(1 and 2), we are left with $|\mathbf{k}_1| = q$ and $|\mathbf{k}_2| = q$ as the residual momenta of P1 and P2 whose least possible value (*say*, q_o) in their G-state would be nothing but the momentum of their zero-point motion.

Since 2.1(a-c) can be applied to any two particles that we pick up in the system, all the $N(N-1)/2$ pairs in the G-state of the system have to have $|\mathbf{K} = \mathbf{0}|$ which implies that all the $N(N-1)/2$ CM points (a CM point is the mid point of the line joining the centers of mass of two chosen particles) of the pairs get fixed in position space which further implies that the position of each particle also gets fixed [9], -of course, with certain amount of position and momentum uncertainty due to wave particle duality.

2.2. G-state(SPB) does not ensure $K = 0$ for every two particles

(a). We note that the energy of the pair is given by

$$E(2) = \frac{\hbar^2}{2m}(k_1^2 + k_2^2) = \frac{\hbar^2}{4m}(k^2 + K^2) = \frac{\hbar^2}{4m}(4q^2 + K^2) \tag{3}$$

which implies that it has its minimum value only if $|\mathbf{q}|$ and $|\mathbf{K}|$ have their minimum values ($|\mathbf{q}| = q_o$ and $|\mathbf{K}| = 0$). However, two particles in the G-state(SPB) do not have $|\mathbf{K}| = 0$ except for the particles having *equal and opposite* $\mathbf{q}$ (*i.e.*, $\mathbf{k}_1 = -\mathbf{k}_2 = \mathbf{q}$) and even for this case SPB theories (as shown in Section 2.3, below) do not prescribe any condition to ensure minimum value for $|\mathbf{q}|$. To this effect, it may be noted that during the process through which a SIB reaches its G-state, *changes in $\mathbf{q}$ can be totally independent of the changes in $\mathbf{K}$*.

As such we find that the energy of G-state(SPB) [*say*, E_o(SPB)] is not ensured for its minimum value and this is evident from the fact that even the single main term of contribution to per particle E_o(SPB) [1, 10, 11], *i.e.*, $4\pi a\hbar^2/mv = ah^2/\pi md^3$ (with $v = d^3$ being the average volume per particle and d the inter-particle distance), is higher than that $(h^2/8md^2)$ in G-state(PPB) [4,5] by a factor of $(8a/\pi d)$ which for liquid helium falls around 2.

(b). Interestingly, when $|\mathbf{K}| = 0$ is applied to all pairs of particles in G-state(SPB) to minimize E_o(SPB) for the energy of their CM motions, we immediately find that particles in each pair

have *equal and opposite* q whose minimum value can not be less than $q_o = \pi/d$ (see Section 2.3(d) below).

2.3. G-state (SPB) does not ensure minimum $|q|$

(a). Following the statement of shape independent approximation (Huang [11], p.279, Section 13.2 of this book), "at low energies the potential acts as if it were a *hard sphere* potential of diameter a" where a represents the s-wave scattering length, it is clear that the volume occupied exclusively by a particle can not be smaller than a^3.

(b). As found experimentally by Grisenti et al. [12] (who also report theoretical a for a comparison), ^4He atoms have $a \approx 100\mathring{A}$ at 1 mK energy, while they have $a \approx 3\mathring{A}$ at $\approx$ 1K energy [10]. This fact indicates that a of the particles of a milli-Kelvin energy is about 30 times larger than that of a Kelvin energy.

(c). In what follows, atoms of different energy/ momentum in G-state(SPB) exclusively occupy different volumes that differ significantly. For example a^3 volume occupied by a ^4He atom of 1 mK energy is estimated to be about 27,000 times that of 1 K energy and it could be even more for a zero momentum particle. As shown below [Section 2.3(d)], we use this observation and minimize the total sum of energies of N particles to prove that particles in the G-state of a SIB have equal energy as depicted by Fig.1(C) rather than different energies as depicted by Fig.1(B) and concluded by SPB theories [1,2,6]

(d). Assuming that i-th atom exclusively occupies a cavity of volume v_i with least possible energy, the ground state energy of the system can be written as

$$E = \sum_i^N \frac{h^2}{8mv_i^{2/3}}, \tag{4}$$

with the condition

$$\sum_i^N v_i = \mathrm{V}. \tag{5}$$

A simple algebra using these two relations reveals that E (in Eqn. 4) has its minimum value only if all v_i are equal to V/N and this renders

$$E_o = \frac{Nh^2}{8md^2} = N\varepsilon_o = N\frac{\hbar^2 q_o^2}{2m} \tag{6}$$

which implies $q_o = \pi/d$. Evidently, E_o(SPB) is also not ensured for the minimum value of the energy contributions from the relative motions of two particles. It is evident that Eqns.(4) and (6) are consistent with an obvious condition known as excluded volume condition [13] which states that each particle such as ^4He atom occupies certain volume exclusively due to its short range HC interaction with other particles. In formulating our theory [4,5] of a SIB like liquid ^4He, we obtain Eqns.(4) and (6) by using our conclusion that two HC particles have to have $\lambda/2 \le d$ [14] (*i.e.*, $q \ge \pi/d$) which is consistent with excluded volume condition [13] as well as with wave packet manifestation of a particle (a consequence of wave particle duality). Not surprisingly, it appears that at low energy the size of wave packet $\lambda/2 = \pi/q = h/2\sqrt{2mE}$

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serves as the effective size of a HC particle and, interestingly, $a \approx 100\mathring{A}$ of ^4He atoms at 1 mK energy and $a \approx 3\mathring{A}$ at about 1K seems to satisfy $E^{-1/2}$ dependence.

3. Other important observations

As evident from [1, 10, 11], SPB theories replace two body *impenetrable* hard core (HC) potential defined by $V_{HC}(r_{ij} > a) = 0$ and $V_{HC}(r_{ij} \leq a) = \infty$ by a *penetrable* $\delta-$potential when they apply perturbative method by using

$$V_{HC}(r_{ij}) \equiv \frac{4\pi a \hbar^2}{m} \delta(r_{ij}). \tag{7}$$

where right hand side potential term does not assume infinitely high value even for $r_{ij} = 0$ since the strength of Dirac delta ($viz.$, $4\pi a \hbar^2/m$) [10, 11] is a small finite value. One may find that no simple logic justifies this equivalence of two physically different potentials, -one having infinitely high value for $r_{ij} = 0$, while the other having only a small finite value. Similarly, the variation approach based on Jastrow or Jastrow-Feenberg function [15, 16] ignores the implication of the shape independent approximation as stated in Section 2.3(a).

4. Concluding remarks

(i) The ground state of a SIB [G-state(SPB)] as conculded by using single particle basis [1] is not a state of least possible energy.

(ii) Subjecting every two particles in G-state(SPB) to the conditions for the minimization of their energy [*viz.* with $|\mathbf{K}| = 0$ and Eqns.(4 and 5) for the energy of residual q], we obtain nothing but G-state(PPB). This not only underlines the accuracy of our theory [4,5] which renders G-state(PPB) but also concludes that one has to have *particle pair basis* as the starting point of the theory in question as we do in [4,5].

(iii) When the particles in a system interact through a two body interaction, a pair of particles forms its natural and logical basic unit. Naturally, theoretical models based on single particle basis can not be expected to reveal results of desired accuracy as shown here for G-state(SPB). In view of the fact that the accuracy of our understanding of the G-state of a system plays an important role in revealing its physical behavior, the present analysis helps in establishing that $p = 0$ condensate does not represent the true form of condensate responsible for the superfluidity of a SIB; as concluded in [4,5], true condensation occurs in a state of pair of particles each having $q = q_o$ and $K = 0$ [4,5].

(iv) In agreement with the results of this study, we find [4,5] that inter-particle HC repulsion decides crucially important G-state of a SIB like liquid ^4He because it serves as the origin of the excluded volume condition [13] which helps in determining the least possible energy of a particle, while inter-particle attraction is found to bind all particles into a single unit (a kind of macroscopic molecule) with a net binding energy which serves as the origin of several important aspects of superfluid SIB [4,5]. The G-state of a SIB has nothing but the potential energy $-V_o$ (representing the flat potential surface on which the particles are free to move in order of their locations [4.5]) added with the energy of zero-point motion which too serves as a potential energy responsible for the zero-point repulsion. It does not have different number of particles

with different momenta (including $p = 0$) as concluded by SPB theories and represented by Fig. 1(B); rather, as depicted in Fig.1(C), each particle has $p = \hbar q_o = h/2d$ momentum and cooresponding zero-point energy, $\varepsilon_o = h^2/8md^2$.

(v). Even in case of a system of non-interacting bosons (SNIB) contained in a box of finite size, L, the G-state is represented exactly by the superposition of two plain waves of equal and opposite momenta (q, -q) [with $q = q_o = \pi/L$ which is not zero], not by a plane wave which implies that each particle has non-zero momentum/energy (indeed of a infinitely small value for $L >> d$). Evidently, there is no condensate of particles with $p = 0$ and the G-state in its description is identical to that concluded in [4,5] for a SIB like liquid ^4He. Having a similar analysis to the particles in a dilute bose gas trapped in a harmonic potential [17], one can easily find that $p = 0$ condensate does not exist in the so called BEC state discovered in 1995 since here too the G-state has particles only with non-zero energy/ momentum.

As such our present analysis unequivocally concludes that the popular belief *that there exists $p = 0$ condensate in the superfluid phase of a SIB such as liquid ^4He and trapped dilute gases* is nothing but a myth and this underlines the potential of our theory [4,5] to explain the behavior of a SIB. .

References

[1] Anderson, J.O.: Rev. Mod. Phys. 76, 599-639 (2004).

[2] Yukalov, V.I., Kleinert, H.: Phys. Rev. A73, 063612(2006).

[3] Ettouhami, A.M.: Reexamining Bogoliubov's theory of an interacting Bose gas, arXiv:cond-mat/ 0703498v1 (2007).

[4] Jain, Y.S.: Macro-orbitals and microscopic theory of a system of interacting bosons, arXiv:cond-mat/0606571.

[5] Jain, Y.S.: New approach to the microscopic theory of a system of interacting bosons -I. Basic foundations and Superfluidity (sent for publication) is a revised version of [4] with additional sections); its pdf copy can be requested from the author.

[6] Bogoliubov, N. N.: J. Phys, USSR, 11, 23-32 (1947).

[7] Pashitskii, E.A., Mashkevich, S.V., Vil'chinskii, S.I. Phys. Rev. Lett. : 89, 075301 (2002); Tomchenko, M.: Low Temp. Phys, 32, 38 (2006),

[8] Chutia, S.: A study of certain properties of superfluid helium-4 based on macro-orbital theory, Ph.D. Thesis (2007), Department of Physics, North-Eastern Hill University, Shillong - 793 022, India.

[9] Fixing the positions of N particles demands fixation of only N points, while fixing of $N(N-1)/2$ CM points fixes more than desired number of points related to the positions of N particles. Since $N(N-1)/2 \geq N$ is true for all $N \geq 3$, fixing of $N(N-1)/2$ CM points naturally ensures fixation of the positions of N particles except for a system of $N = 2$ which has only 1 CM point; this rightly agrees with the fact that one can not fix the positions of two particles or their 6 degrees of freedom by fixing the position of only 1 CM point, -equivalent to fixing only 3 degrees of freedom leaving 3 degrees of freedom to remain unfixed and these are two rotations and one vibration. We also note that the coordinates of each CM point are related to the position coordinates of two chosen particles, positions of all the $N(N-1)/2$ CM points get automatically fixed if we fix the positions of N particles.

[10] Pathria, R.K.: Statistical Mechanics, Pergamon Press Oxford (1976).

[11] Huang, K.: Statistical Mechanics, Wiley Eastern Limited, New Delhi (1991).

[12] Grisenti et. al.: Phys. Rev. Lett. 85, 2284-2287 (2000) .

[13] Kleban, P.: Phys. Letters A 49, 19-20(1273).

[14] Jain, Y.S. : Cent. Euro. J. Phys. 2, 709-719(2004).

[15] C. -W. Woo in The Physics of liquid and solid helium, K.H. Benneman and J.B. Ketterson (Editors), Part-I, Wiley, New York (1976).

[16] Feenberg, F.: The theory of Quantum Fluids, Academic, New York, 1969).

[17] Dalfovo, F., Giorgini, S. Pitaevskii, L.P. and Stringari, S.: Rev. Mod. Phys. 71 463-512 (1999); arXiv/Cond-mat/9806038.
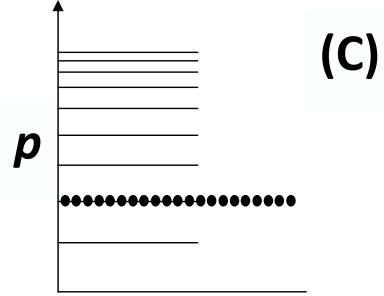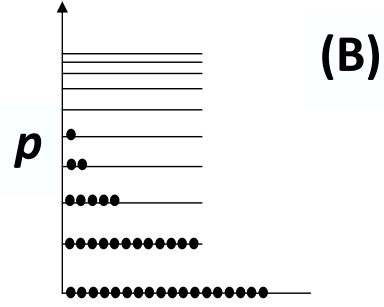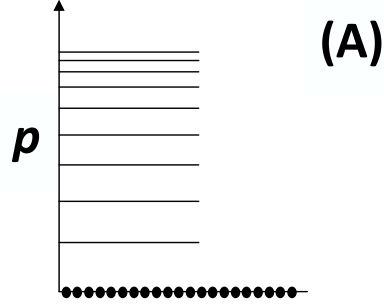
Fig.1 : Schematic of distribution of N bosons in their ground state. (A) All the N particles occupy $p = 0$ state in a system of non-interacting bosons, (B) depletion of $p = 0$ condensate (*i.e.* only a fraction of N occupy $p = 0$ state) in weakly interacting boson system as predicted by Bogoliubov model [6], and (C) all the N particles occupy a state of $q = \pi/d$ and $K = 0$ as concluded from our recent theory [4,5].